

ON CHARACTERIZATIONS OF PARETO AND  
WEIBULL DISTRIBUTIONS BY CONSIDERING  
CONDITIONAL EXPECTATIONS OF  
UPPER RECORD VALUES

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ABSTRACT. Let  $\{X_n, n \geq 1\}$  be a sequence of i.i.d. random variables with absolutely continuous cumulative distribution function(cdf)  $F(x)$  and the corresponding probability density function(pdf)  $f(x)$ . In this paper, we give characterizations of Pareto and Weibull distribution by considering conditional expectations of record values.

## 1. Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function(cdf)  $F(x)$  and probability density function (pdf)  $f(x)$ . Let  $Y_n = \max\{X_1, X_2, \dots, X_n\}$  for  $n \geq 1$ . We say  $X_j$  is an upper record value of this sequence, if  $Y_j > (<)Y_{j-1}$  for  $j > 1$ . The indices at which the upper record values occur are given by the record times  $\{U(n), n \geq 1\}$ , where  $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$  with  $U(1) = 1$ . We assume that all upper record values  $X_{U(i)}$  for  $i \geq 1$  occur at a sequence  $\{X_n, n \geq 1\}$  of i.i.d. random variables.

Lee(2003) showed that  $X \in PAR(\theta)$  if and only if  $(\theta + 1)^i E[X_{U(n+i)} \mid X_{U(m)} = y] = \theta^i E[X_{U(n)} \mid X_{U(m)} = y]$  for  $i = 1, 2, 3, n \geq m + 1$ . Also, Dembińska & Wesolowski(2000) obtained characterizations of the exponential, Pareto, and power distributions by considering this problem with  $E(R_{m+k} \mid R_m) = aR_m + b$ . Further, Ahsanullah and Shakil(2012)

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proved that  $X \in PAR(\beta)$  if and only if  $(\delta - k)E[X_{U(n+1)}^k | X_{U(m)} = y] = \delta E[X_{U(n)}^k | X_{U(m)} = y], m \geq n + 1$ .

In this paper, we give the characterizations of Pareto and Weibull distributions by considering conditional expectations of record values.

**2. Main results**

**THEOREM 2.1.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent identically distributed random variables with cumulative distribution function  $F(x)$  which is absolutely continuous with pdf  $f(x)$  and  $E(X^k) < \infty$ . For positive integers  $k, l$  and a positive real number  $\theta < k$ ,*

(2.1)  $(\theta - k)^l E[X_{U(n+l)}^k | X_{U(m)} = y] = \theta^l E[X_{U(n)}^k | X_{U(m)} = y], n \geq m + 1$

if and only if

(2.2)  $F(x) = 1 - cx^{-\theta}, 0 < c \leq x, \theta > 0.$

*Proof.* For necessary part, it is easy to see that (2.2) implies (2.1).

For the sufficiency part, using Ahsanullah formula(1995), we get the following equation

(2.3) 
$$\begin{aligned} & \frac{(\theta - k)^l}{(n + l - m - 1)!} \int_y^\infty \left( - \ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{n+l-m-1} x^k f(x) dx \\ &= \frac{\theta^l}{(n - m - 1)!} \int_y^\infty \left( - \ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{n-m-1} x^k f(x) dx. \end{aligned}$$

Since  $F(x)$  is absolutely continuous, we can differentiate  $(n - m)$  times both sides of (2.3) with respect to  $y$  and simplify to obtain the following equation

(2.4) 
$$\frac{1}{\bar{F}(y)\Gamma(l)} \int_y^\infty \left( - \ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{l-1} x^k f(x) dx = \frac{\theta^l}{(\theta - k)^l} y^k.$$

By setting  $s = \frac{\bar{F}(x)}{\bar{F}(y)}$  in (2.4) we get

$$\frac{1}{\Gamma(l)} \int_0^1 [\bar{F}^{-1}(s\bar{F}(y))]^k (- \ln(s))^{l-1} ds = \frac{\theta^l}{(\theta - k)^l} y^k.$$

Setting  $\bar{F}(y) = t$ , we find that

$$\frac{1}{\Gamma(l)} \int_0^1 [\bar{F}^{-1}(st)]^k (- \ln(s))^{l-1} ds = \frac{\theta^l}{(\theta - k)^l} [\bar{F}^{-1}(t)]^k, 0 < t < 1.$$

Dividing both sides by  $\frac{(\theta-k)^l}{\theta^l}$  and substituting once again  $s = e^{-v}$  and  $t = e^{-w}$ , we have

$$\frac{(\theta - k)^l}{\theta^l \Gamma(l)} \int_0^\infty [\bar{F}^{-1}(e^{-(v+w)})]^k (v)^{l-1} e^{-v} dv = [\bar{F}^{-1}(e^{-w})]^k, \quad 0 < w < \infty.$$

Now we use applications of integrated Cauchy functional equation in C.R. Rao & D.N. Shanbhag(1994) and let  $H(w) = [\bar{F}^{-1}(e^{-w})]^k$ . Consequently,

$$(2.5) \quad \int_{R^+} H(v + w)\mu(dv) = H(w), \quad 0 < w < \infty,$$

where  $\mu$  is a finite measure on  $R^+$ , which is absolutely continuous with respect to the  $[L]$  measure and is defined by

$$(2.6) \quad \mu(dv) = \frac{(\theta - k)^l}{\theta^l \Gamma(l)} (v)^{l-1} e^{-v} dv.$$

Observe that  $H$  is strictly increasing on  $[0, \infty)$  since it is a superposition of two strictly decreasing functions. Hence, since  $H$  is continuous it follows that

$$(2.7) \quad H(w) = \begin{cases} \gamma + \alpha(1 - \exp(\eta w)) & \text{if } \eta \neq 0, \\ \gamma + \beta w & \text{if } \eta = 0, w > 0, \end{cases}$$

where  $\alpha, \beta, \gamma$  are all constants and  $\eta$  is determined such that

$$(2.8) \quad \int_{R^+} \exp(\eta x)\mu(dx) = 1.$$

From (2.8) we get

$$(2.9) \quad 1 = \frac{(\theta - k)^l}{\theta^l \Gamma(l)} \int_0^\infty (x)^{l-1} e^{(\eta-1)x} dx.$$

which means that  $\eta < 1$ , since the integral at the right-hand side has to converge. Consequently,

$$(2.10) \quad \frac{(\theta - k)^l}{\theta^l} = (1 - \eta)^l.$$

From equation (2.10) we obtain  $\eta = \frac{k}{\theta}$  and it follows that  $0 < \eta < 1$ . Using equation (2.7) we get

$$[\bar{F}^{-1}(e^{-x})]^k = \gamma + \alpha(1 - e^{\eta x}).$$

By applying Theorem 1 in Anna Dembińska & Jacek Wesolowski(2000) for  $a = \frac{\theta^l}{(\theta-k)^l} > 1$  and  $\alpha + \gamma = 0$ , we have

$$\bar{F}(z) = cz^{-\theta}, \quad 0 < c \leq z, \quad \theta > 0, \quad c = \gamma^{\frac{\theta}{k}}.$$

Hence, the theorem is proved. □

**THEOREM 2.2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of independent identically distributed random variables with cumulative distribution function  $F(x)$  which is absolutely continuous with pdf  $f(x)$ . For positive integers  $k, l$  and a positive real number  $\theta$ ,*

(2.11) 
$$E[X_{U(n+l)}^k | X_{U(m)} = y] = E[X_{U(n)}^k | X_{U(m)} = y] + l\theta^k, \quad n \geq m + 1.$$

if and only if

(2.12) 
$$F(x) = 1 - e^{-(\frac{x}{\theta})^k}, \quad 0 \leq x, \quad \theta > 0.$$

*Proof.* For necessary part, it is easy to see that (2.12) implies (2.11). For the sufficiency part, we get

(2.13) 
$$\begin{aligned} & \frac{1}{(n+l-m-1)!} \int_{-\infty}^y \left( -\ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{n+l-m-1} x^k f(x) dx \\ &= \frac{1}{(n-m-1)!} \int_{-\infty}^y \left( -\ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{n-m-1} x^k f(x) dx + l\theta^k. \end{aligned}$$

Since  $F(x)$  is absolutely continuous, we can differentiate  $(n-m)$  times both sides of (2.13) with respect to  $y$  and simplify to obtain the following equation

(2.14) 
$$\frac{1}{\bar{F}(y)\Gamma(l)} \int_y^\infty \left( -\ln \frac{\bar{F}(x)}{\bar{F}(y)} \right)^{l-1} x^k f(x) dx = y^k + l\theta^k.$$

By setting  $s = \frac{\bar{F}(x)}{\bar{F}(y)}$  in (2.14) we get

$$\frac{1}{\Gamma(l)} \int_0^1 [\bar{F}^{-1}(s\bar{F}(y))]^k (-\ln(s))^{l-1} ds = y^k + l\theta^k.$$

Setting  $\bar{F}(y) = t$ , we find that

$$\frac{1}{\Gamma(l)} \int_0^1 [\bar{F}^{-1}(st)]^k (-\ln(s))^{l-1} ds = [\bar{F}^{-1}(t)]^k + l\theta^k, \quad 0 < t < 1.$$

By setting  $s = e^{-v}$  and  $t = e^{-w}$ , we obtain

$$\frac{1}{\Gamma(l)} \int_0^\infty [\bar{F}^{-1}(e^{-(v+w)})]^k (v)^{l-1} e^{-v} dv = [\bar{F}^{-1}(e^{-w})]^k + l\theta^k, \quad 0 < w < \infty.$$

In the same way as the proof of Theorem 1.1, we get

$$(2.15) \quad 1 = \frac{1}{\Gamma(l)} \int_0^\infty (x)^{l-1} e^{(\eta-1)x} dx,$$

which means that  $\eta < 1$ , since the integral at the right-hand side has to converge. Consequently,

$$(2.16) \quad 1 = (1 - \eta)^l.$$

From equation (2.16) we obtain  $\eta = 0$ .

Using equation (2.7) we get

$$[\bar{F}^{-1}(e^{-x})]^k = \gamma + \beta x.$$

By applying Theorem 1 in Anna Dembińska & Jacek Wesolowski(2000) for  $\gamma = 0$ ,  $a = 1$  and  $\beta = \theta^k$ , we have

$$\bar{F}(z) = e^{-(\frac{z}{\theta})^k}, \quad z \geq 0, \quad \theta > 0.$$

Hence, the theorem is proved.  $\square$

### References

- [1] M. Ahsanullah, *Record Statistics*, **Inc, Dommack NY, 1995**.
- [2] M. Ahsanullah and M. Shakil, *A note on the characterizations of the pareto distribution by upper record values*, Commun. Korea Math. soc. **27** (2012), 835-842.
- [3] A. Dembińska and J. Wesolowski, *Linearity of regression for non-adjacent record values*, J. Statist. Plann. Inference **90** (2000), 195-205.
- [4] M. Y. Lee, *Characterizations of the pareto distribution by conditional expectations of record values*, Commun. Korea Math. soc. **18** (2003), 127-131.
- [5] C. R. Rao and D. N. Shanbhag, *Choquet-Deny Type Functional Equations with Applications to Stochastic Models*, **Wiley, NY, 1994**.

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